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A MIXED VARIATIONAL PRINCIPLE AND ITS APPLICATION TO THE NONLINEAR BENDING PROBLEM OF ORTHOTROPIC TUBES—II. APPLICATION TO NONLINEAR BENDING OF CIRCULAR CYLINDRICAL TUBES

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Abstract—A procedure for deriving mixed variational principles for nonlinear shell analysis was presented in Part I and formulated in more detail for shells of weak curvatures and for circular cylindrical shells. The cylindrical shell principle is extended here to accommodate those special cases in which the retention of mixed terms in the compatibility equations is of significance. Emphasis is put on the interaction of longitudinal extensional strains with large circumferential changes of curvature. As a simple first example, it is applied to the orthotropic Brazier process, for which an Euler–Lagrange equation and perturbation solution are also derived. Variational principles have important uses for providing direct, approximate "engineering solutions" to highly nonlinear problems. Such solutions complement the more exact but cumbersome finite element or double series techniques. In the present case, reasonable simplifying assumptions are introduced in the extended principle to construct approximate principles and equations for the problem of strong, nonlinear, nonuniform bending of finite-length orthotropic tubes. As a specific example, the pure bending of a clamped, finite-length tube is studied. Approximate analysis is carried to collapse (or local buckling). Some of the local buckling results are compared with numerical results from the literature.

1. THE EXTENDED VARIATIONAL PRINCIPLE

The need for modifying the variational functional arises in problems involving highly deformed shells with strong directional effects in which the largest curvature changes are of the order $O(1/R)$, while the smallest are of the order $O(\varepsilon/R)$. In the case of nonlinear bending of relatively long cylindrical shells, the nondimensional circumferential curvature change ak_{ss} might be $O(1)$ while the longitudinal curvature changes ak_{xx} are of the order of the middle-surface extensional strains e_{xx} caused by the bending. Small mid-surface extensional strains are assumed, so that any power of these strains of order greater than unity is suppressed.

The reference configuration is a circular cylindrical shell of radius a, wall thickness t, length L and centroidal area moment of inertia $I = \pi a^3 t$. The coordinates on the middle surface are x along the generator and $s = \theta a$ on the middle surface in the circumferential direction. These are also Lagrangian (material) coordinates on the deformed middle surface. All tensorial operations refer to the undeformed metric, which is Cartesian ($a_{xx} = a_{ss} = 1$, $a_{xs} = 0$). The Codazzi compatibility equations, with the relevant mixed terms retained, are

$$k_{ss,x} - k_{xs,s} - \left(\frac{1}{a}\right) e_{ss,x} = 0 \tag{1a}$$

$$k_{xx,s} - k_{xs,x} - \left(\frac{1}{a} + k_{ss}\right) \lambda = 0, \tag{1b}$$

where $\lambda = e_{xx,s} - \gamma_{xs,x}$ is the geodesic curvature of the deformed generators. Obviously, the term $k_{ss}e_{ss}$ not shown in eqn (1a) is $O(\varepsilon)$ compared to k_{ss} and has been neglected. However, if ak_{xx} is $O(\varepsilon)$ and ak_{ss} is $O(1)$, then the mixed term $k_{ss}\lambda$ in eqn (1b) is of the same order as $k_{xx,s}$ and should not be suppressed. It is this special situation which requires the modification of the variational functional.

The curvature function is introduced by $k_{xs} = \psi_{,xs}$; see also eqn (13b) in Part I. Dropping nonlinear extensional strain terms, the Codazzi and Gauss compatibility equations become

$$k_{ss} = \psi_{,ss} + \frac{1}{a} e_{ss} \tag{2a}$$

$$(k_{xx} - \psi_{,xx})_{,s} = \left(\frac{1}{a} + \psi_{,ss}\right)\lambda \tag{2b}$$

$$\left(\frac{1}{a} + \psi_{,ss}\right)k_{xx} + \lambda_{,s} + e_{ss,xx} - \psi_{,xs}^2 = 0. \tag{2c}$$

Now new variables $\bar{\theta}$ and $\bar{r}_s$ are defined by

$$\bar{\theta} = \theta + \psi_{,s} = (s/a) + \psi_{,s}, \quad \bar{r}_s = (\bar{\theta}_{,s})^{-1},$$

such that for any function F,

$$F_{,s} = \bar{\theta}_{,s} F_{,\bar{\theta}}; \quad F_{,x\bar{\theta}} - F_{,\bar{\theta}x} = \bar{r}_s \psi_{,ssx} F_{,\bar{\theta}}.$$

Upon change of variables, eqns (2b, c) can be reduced to

$$\lambda = (k_{xx} - \psi_{,xx})_{,\bar{\theta}} \quad \text{(Codazzi)} \tag{3a}$$

$$\lambda_{,\bar{\theta}\bar{\theta}} + \lambda - [\bar{r}_s(\psi_{,xs}^2 - e_{ss,xx}) - \psi_{,xx}]_{,\bar{\theta}} = 0 \quad \text{(Gauss)}. \tag{3b}$$

It is noted that for small deformations, $\bar{\theta}$ is the deformed circumferential angular slope and $\bar{\theta}_{,s}$ is the deformed circumferential curvature.

Using the modified Gauss equation, one can express the enhanced variational Π_2 [Part I, eqn (17)] as

$$\Pi_2(e_{\alpha\beta}, \psi, f) = \iint_A \{U - f\bar{\theta}_{,s}\{\lambda_{,\bar{\theta}\bar{\theta}} + \lambda - [\bar{r}_s(\psi_{,xs}^2 - e_{ss,xx}) - \psi_{,xx}]_{,\bar{\theta}}\}\}\,\mathrm{d}A + P. \tag{4}$$

The use of $(\bar{\theta}_{,s}, f)$ as multiplier is for convenience only. To eliminate $e_{\alpha\beta}$ from the functional, the coefficients of $\delta e_{\alpha\beta}$ in $\delta\Pi_2$ are set equal to zero with the following results:

$$\left.\frac{\partial U}{\partial e_{xx}}\right|_{\psi} = n^{xx} = -[(f_{,\bar{\theta}\bar{\theta}} + f)\bar{\theta}_{,s}]_{,s} \tag{5a}$$

$$\left.\frac{\partial U}{\partial \gamma_{xs}}\right|_{\psi} = n^{xs} = [(f_{,\bar{\theta}\bar{\theta}} + f)\bar{\theta}_{,s}]_{,x} \tag{5b}$$

$$\left.\frac{\partial U}{\partial e_{ss}}\right|_{\psi} = n^{ss} = -f_{,\bar{\theta}xx}. \tag{5c}$$

For more details on the constitutive relations, see Section 2.2 of Part I. In general, $n^{\alpha\beta} = n^{\alpha\beta}(f, \psi)$. Proceeding as in Part I, $e_{\alpha\beta}(f, \psi)$ are introduced into Π_2 and partial integration is performed, to arrive at the mixed potential $\Pi^*(f, \psi)$ as follows:

$$\Pi^* = \iint_A \left[U_m^* + \frac{1}{a} f_{,\theta xx}\psi - \tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(f_{,\theta})_{,\alpha\beta}\psi_{,\gamma}\psi_{,\delta} \right] \mathrm{d}A + P^*, \tag{6}$$

where, for simplicity, $p^\alpha = 0$ was taken. The remarks of Part I on the Π^* principle regarding the admissible fields and P^* also apply here. In particular, $P^* = 0$ for cases in which the only nonhomogeneous quantities are applied force resultants on ∂L_n. This includes the common case of applied beam-type axial bending moments and shear forces on the ends of a tube. The Euler–Lagrange equations and associated boundary conditions are not affected by the extended formulation. The main effect is on the relations $n^{\alpha\beta}(f, \psi)$ and $k_{\alpha\beta}(f, \psi)$, which are now more complicated.

In problems involving a large ratio of "differentiation lengths" ($\mu = L_x/L_s \gg 1$), the $e_{ss,xx}$ and $\psi_{,xs}^2$ terms can be neglected in Π_2, thus leading to a simplified form for Π^*:

$$\Pi^* = \iint_A (U_m^* + f_{,xxs}\psi)\, \mathrm{d}A + P^*. \tag{7}$$

The nonlinearities in Π^* are contained in U_m^* through the expressions for $n^{\alpha\beta}$ and k_{xx}. Since e_{ss} is small, it is also removed from the energy and from k_{ss}. Some of the local edge effects at the ends of the tube (where μ is small) are lost. The decision on whether to use eqn (6) or eqn (7) depends on the problem at hand.

The simplification in eqn (7) is related to the "semi-membrane" approximation. For further details on this approximation and its uses in cylindrical shell theory, see Calladine (1983) and Axelrad (1985a), where more references are given.

2. NONLINEAR BENDING OF CIRCULAR CYLINDRICAL TUBES

2.1. *Pure bending of infinitely long tubes*

Brazier (1927) analyzed the deformation and collapse of an isotropic tube using small deformation theory. Hayashi (1949) extended the results to orthotropic tubes. Many studies have been made since then and stronger nonlinearities introduced. In this section, the mixed variational principle is applied to a strongly nonlinear formulation of the problem in terms of either direct methods or an Euler–Lagrange equation. Although the main purpose of this section is to demonstrate the use of the principle, some of the results will be useful for the nonuniform case.

Since all quantities are now independent of axial coordinate x, eqns (3a, b) can be reduced to

$$\begin{aligned} k_{xx,\bar\theta} &= \lambda \quad (\lambda = e_{xx,s}) \\ \lambda_{,\theta\theta} + \lambda &= 0, \end{aligned} \tag{8}$$

with solution

$$\begin{aligned} \lambda &= \kappa\cos\bar\theta + \kappa_1 \sin\bar\theta \\ k_{xx} &= \kappa\sin\bar\theta - \kappa_1\cos\bar\theta. \end{aligned} \tag{9}$$

The constants κ and κ_1 are associated with the imposed bending. For bending in the xy-plane, $\kappa_1 = 0$. Let $\bar y$ be the deformed y-coordinate of a material point. Then

$$\begin{aligned} \cos\bar\theta &\cong \bar y_{,s} \\ e_{xx} &= \kappa\bar y + c. \end{aligned} \tag{10}$$

Here, c is associated with axial loads and is zero for pure bending. Thus, the satisfaction of the compatibility equations is *equivalent* to the requirement that e_{xx} be proportional to $\bar{y}$. The latter can be deduced directly from considerations of symmetry, so that this implicitly validates the solution scheme.

The effects of e_{ss} in tube bending problems are small (compared with those of e_{xx} and k_{ss}) and it is usually neglected in tube bending analysis. Also, since $\kappa_{xx} = O((1/a)e_{xx})$, its effects on the strain energy are small compared with the corresponding extensional energy terms due to e_{xx}, and it can be omitted there. The constitutive relations can be used to express n^{xx} in terms of e_{xx}. Up to this point no assumption has been made regarding these relations, but at this point a linear orthotropic material is assumed, so that

$$n^{xx} = E_x t e_{xx} = E_x t \kappa \bar{y}, \tag{11}$$

where E_x is the extensional modulus in the x-direction. The equilibrium requirement $M = \oint n^{xx}\bar{y}\,\mathrm{d}s$ leads to

$$n^{xx} = \frac{Mt}{\bar{I}}\bar{y}, \quad \kappa = \frac{M}{E_x\bar{I}}, \quad \bar{I} = \oint \bar{y}^2 t\,\mathrm{d}s. \tag{12}$$

Note that these are, in essence, beam-type formulas, with κ being the "tube-beam" curvature. Also, $\bar{I}$ is the moment of inertia of the deformed cross-section and D_s is the circumferential bending rigidity. With the above in mind, the mixed energy becomes:

$$\frac{1}{L}\Pi^*(\psi) = \oint \frac{D_s}{2}\psi_{,ss}^2\,\mathrm{d}s - \frac{M^2}{2E_x\bar{I}}. \tag{13}$$

If $E_x t$ varies (symmetrically) with s, then $E_x\bar{I}$ should be replaced by $\oint E_x t\bar{y}^2\,\mathrm{d}s$. To determine $\bar{I}$, first $\bar{y}$ is obtained from ψ by

$$\bar{y} = \int_0^\theta \cos(\theta + \psi_{,s})a\,\mathrm{d}\theta \quad (\theta = s/a). \tag{14}$$

For use in direct methods, ψ is expressed in a series form, such as $\psi = -a\Sigma_1^N \Gamma_n \cos 2n\theta$. Also, $\bar{I}$ and $\bar{y}$ are calculated from eqns (12) and (14) and Γ_i are obtained from $\partial\Pi^*/\partial\Gamma_i = 0$. The leading term ($N = 1$) is dominant and is exact for small deformations, as was shown by Brazier (1927) and others. Calladine (1983) used it for moderate rotation analysis. For $N = 1$, the expression for $\bar{I}$ is

$$\bar{I}/I = 1 - 2\Gamma - \tfrac{8}{9}\Gamma^2 + \tfrac{28}{9}\Gamma^3 + \tfrac{32}{75}\Gamma^4 + O(\Gamma^5). \tag{15}$$

The use of the first two terms only yields Brazier's classical result.

A second approach is to obtain from $\delta\Pi^* = 0$, an exact Euler–Lagrange equation for the problem. From eqn (14),

$$\psi_{,ss} = \mp \frac{\bar{y}_{,ss}}{(1-\bar{y}_{,s}^2)^{1/2}} - \frac{1}{a}. \tag{16}$$

Introducing the above result into $\delta\Pi^* = \oint[(D_s/2)\delta(\psi_{,ss}^2) + (E_x t\kappa^2/2)\bar{y}\delta\bar{y}]\,\mathrm{d}s = 0$, performing the modified integrations by parts, and setting equal to zero the multiplicand of $\delta\bar{y}$, one obtains

$$\{(1-\bar{y}_{,s}^2)^{-1/2}[D_s(1-\bar{y}_{,s}^2)^{-1/2}\bar{y}_{,ss}]_{,s}\}_{,s} + \kappa^2 E_x t\bar{y} = 0. \tag{17}$$

Solutions $\bar{y}(\theta, \kappa^2)$ are subject to conditions of periodicity. They can be used to calculate

$\bar{I}(\kappa^2)$ and $M = E_x \kappa \bar{I}(\kappa^2)$. The first maximum of the latter is the collapse moment. It is noted that both D_s and E_x may vary with s.

For $\kappa = 0$, eqn (17) is satisfied by the undeformed shape $\bar{y} = y = a \sin\theta$. To obtain a perturbation series solution for $\bar{y}$ about $\kappa = 0$ (with perturbation parameter η), one sets

$$\bar{y} = a\sin\theta + \sum V_m \eta^m\,; \quad \psi_{,ss} = \sum \psi_{m,ss}\eta^m\,; \tag{18}$$

where

$$\eta = \kappa^2 E_x t a^4/12D_s.$$

These expressions are introduced into eqns (16) and (17). Solutions for V_m and ψ_m are obtained by standard procedures. The process involves quadratures only. Results for the first and second perturbations are:

$$V_1 = -a\sin^3\theta, \qquad \psi_{1,ss} = (3/a)\cos 2\theta$$

$$V_2 = a(-\tfrac{31}{12}\sin^3\theta + \tfrac{2}{5}\sin^5\theta), \quad \psi_{2,ss} = \frac{1}{a}(2\cos 2\theta + \tfrac{5}{4}\cos 4\theta).$$

These can be substituted into $\bar{I}$ to obtain the $M(\kappa)$ response to the second order in N. The first perturbation is identical to the classical solution. The $m = 2$ terms involve additional trigonometric functions, but the deformation remains symmetric. Even though higher order terms were not calculated, it appears that "for all practical purposes" the nonlinear deformation *may be taken to be symmetric*. This does not hold near collapse or buckling, but in those cases the emerging axial nonuniformity renders the present equations invalid.

An alternative formulation involving two nonlinear differential equations in terms of a stress function and the rotation was obtained by Reissner (1961). It was solved by Reissner and Weinitschke (1963) using perturbation and integral equation techniques. The analysis, which was confined to the homogeneous isotropic case, provides firm support for the Brazier approximation.

2.2. *Nonlinear nonuniform bending of finite-length tubes*

Let $\lambda_B = \kappa(x)\cos\bar{\theta}$ be the symmetric solution of the equation $\lambda_{,\theta\theta} + \lambda = 0$, where now κ depends on x (in the above, B stands for "beam"). Also, let λ_s (S stands for "shell") be a solution of the *complete* compatibility equation. Then, $(\lambda_B + \lambda_S)$ is also a solution of the *complete* compatibility equation. This implies that λ_B can be *separated out* from the equation. A similar conclusion is reached by noting that in tube bending problems, $F_{,x} \ll F_{,s}$ for any shell quantity. If a scaling is introduced by $\zeta = \mu x$, such that $F_{,\zeta\zeta} = 0(F_{,ss}, F)$, then $\mu^2 \ll 1$. Hence, the compatibility equation can be cast in the form

$$\lambda_{,\theta\theta} + \lambda + \mu^2(\psi_{,\zeta\zeta s} + \cdots) = 0. \tag{19}$$

Putting $\lambda = \lambda_B + \mu^2\lambda_s + O(\mu^4)$ yields the perturbation equations

$$\begin{aligned} &\lambda_{B,\theta\theta} + \lambda_B = 0 \\ &\lambda_{S,\theta\theta} + \lambda_S + (\psi_{,\zeta\zeta s} + \cdots) = 0, \end{aligned} \tag{20}$$

which implies that λ_B is also the *first order* solution of the compatibility equation, and λ_S is the solution of the *second order* complete equation. Note that λ_B provides the major nonlinear response to bending away from the edges, but λ_S is needed for the boundary conditions and second order corrections. In fact, it is reasonable to suppress the nonlinearities and replace $\bar{\theta}$ with θ for the λ_S solution.

It would be useful if the B and S systems were partitioned in the strain energy too, so that mixed terms would vanish in $\iint U\,dA$. This occurs, of course, in linear theory but also in several nonlinear applications. Tube bending is included in this category† since the bending strains of the B system are *odd* in θ, while the cross-sectional distortion ψ and its S system are *even* in θ. In this case, the stresses of the S system do not contribute to overall equilibrium. The external moments and shear forces are carried entirely by the B system, which can be treated separately.

Integration of λ_B results in

$$E_{xx_B} = e_{xx_B} - \int \gamma_{xs_B,x}\, \mathrm{d}s = \kappa(x)\bar{y} + c(x), \tag{21}$$

where $c(x) = 0$ for bending with symmetrical distortion. If n^{xs} and γ_{xs} are proportional to the external shearing force Q, then for edge loaded beams, $\gamma_{xs_B,x} = 0$. In the more general case of variable shear forces (which should include the case of distributed loading), the equation

$$\left(\frac{n^{xx}}{E_x t}\right)_{,ss} + \mu^2 \left(\frac{n^{xx}}{Gt}\right)_{,\zeta\zeta} = \kappa(x)(\cos\bar{\theta})_{,s} \tag{22}$$

holds for orthotropic materials (equilibrium equations were used for its derivation). Hence

$$e_{xx_B} = \kappa(x)\bar{y} + c(x) + O\left(\mu^2 \frac{E_x}{G}\right). \tag{23}$$

The last term is small compared with the others and can usually be neglected. Considerations of equilibrium yield, as in the uniform bending case,

$$n^{xx} = \frac{Mt}{\bar{I}}\bar{y}; \quad \kappa(x) = \frac{M(x)}{E_x \bar{I}}$$

$$n^{xs} = -\frac{\partial}{\partial x}\left[\frac{M}{\bar{I}} \int t\bar{y}\, \mathrm{d}s\right] \cong -\frac{Q\bar{S}}{\bar{I}}, \tag{24}$$

where

$$\bar{S} = at \int_{\pi/2}^{\theta} \bar{y}\, \mathrm{d}\theta.$$

The neutral surface is chosen such that $\oint \bar{y}\, \mathrm{d}s = 0$. For symmetrical distortions it does not shift. The normal curvature is given by $k_{xx} = \kappa(x)\sin\bar{\theta}$. Noting that the geodesic curvature is $\lambda_B = \kappa(x)\cos\bar{\theta}$, it follows that these are components of the curvature $\kappa(x)$ of the tube as a beam. The rotation of cross-sections β is such that $\kappa(x) = \beta_{,x}$. The complementary energy of the B system is

$$U^*_{mB} = -\int \left(\frac{M^2}{2E_x \bar{I}} + \frac{Q^2 \bar{R}}{2G}\right) \mathrm{d}x, \tag{25}$$

where

$$\bar{R} = \oint \bar{S}^2\, \mathrm{d}s / t\bar{I}^2.$$

† See Section 2.1 for details on the preservation of symmetry in the nonlinear range.

It follows that the B system describes the behavior of the tube as a beam, with beam properties related to the *deformed configuration*. The latter, however, is not yet known since the distortion ψ can be obtained only from the full solution, which includes the S system.

2.3. *Approximate analysis*

In order to simplify the problem, approximations are made (some of these were already discussed before and will be repeated here for completeness).

(1) The semi-membrane approximation is adopted. Thus, the effects of e_{ss} are suppressed. Also, the terms with k_{xx} and k_{xs} are deleted from the strain energy. The more restrictive assumptions of "flexible shell theory" and thin-walled beam theory (Vlasov, 1961; Gjelsvik, 1981), which also neglect the shear strain γ_{xs}, are not made at this stage. The present work goes along with more recent approaches which retain the shear strain effects. See Axelrad (1985a) for more details.

(2) Nonlinear terms are suppressed in the S system. Thus, $\bar{\theta}$ is replaced by θ and the $\psi_{,xs}^2$ term is dropped.

(3) Symmetry of the cross-sectional distortion will be assumed, thus facilitating the partitioning of the B and S systems. This is exact in the case of small deformations, as shown by Brazier (1927), and is the dominant feature even in strong nonlinearities, as long as buckling or collapse do not take place. See Section 2.1 for details.

(4) Moments M and shear forces Q may be applied to the tube boundaries $x = 0, L$. Otherwise, the boundary conditions are homogeneous. Distributed external loading (if applied) may affect the moment and shear distribution but its local effect (as surface loads) is ignored.

(5) The material is linear orthotropic with axes of symmetry in the x, s-directions. Here D_s, E_x and G are the circumferential bending rigidity, extensional and shear moduli, respectively.

With these approximations and assumptions, Π^* becomes

$$\Pi^* = \iint_A \left[\tfrac{1}{2} D_s k_{ss}^2 - \frac{1}{2E_x t}(n^{xx})^2 - \frac{1}{2Gt}(n^{xs})^2 + f_{,sxx}\psi \right] \mathrm{d}s\,\mathrm{d}x - \int_0^L \left(\frac{M^2}{2E\bar{I}} + \frac{Q^2 \bar{R}}{2G} \right) \mathrm{d}x, \tag{26}$$

with

$$k_{ss} = \psi_{,ss}; \quad an^{xx} = -(f_{,\theta\theta} + f)_{,s}; \quad an^{xs} = (f_{,\theta\theta} + f)_{,x}.$$

All the quantities in the surface integral are those of the S system, but the subscript s is omitted for simplicity. If the shear strains are neglected, then the terms with G are dropped. The field variables are ψ, f and any indeterminacy parameters K_i which may be contained in M and Q (such as a fixed-end moment in a statically indeterminate beam-tube). The nonlinearity is contained in the dependence of $\bar{I}$ and $\bar{R}$ on $\bar{y}$ (and, through it, on ψ). Admissible f must satisfy any boundary conditions on ∂L_n. The suppression of k_{xx} and $k_{x\theta}$ in U_m^* precludes the assignment of boundary conditions related to shell rotations or normal displacements on the boundary.

A reasonable solution procedure would be to represent the variables as trigonometric series in θ with x-dependent coefficients,

$$\psi = -a\sum \Gamma_m(x)\cos m\theta, \quad f = a\sum f_m(x)\sin m\theta \quad (m = 2, 4, \cdots), \tag{27}$$

and use the Rayleigh–Ritz process to either obtain an approximate solution by direct means or obtain a system of ordinary differential equations. The problem can, however, be greatly simplified by taking only the leading terms

$$\psi = -a\Gamma_2(x)\cos 2\theta, \quad f = a^2 f_2(x)\sin 2\theta \tag{28}$$

as *approximations in the Rayleigh–Ritz sense*, and use a variational technique to obtain the relevant equations. Brazier (1927) showed that this is exact for small deformations and pure bending of *long shells*. It has been shown to be (approximately) valid well into the nonlinear range. For further discussion, see Reissner and Weinitschke (1963), Antonenko (1981), Calladine (1983), Axelrad (1985a) and others. Substitution into Π^* and denoting $F' = \mathrm{d}F/\mathrm{d}x$, yield:

$$\Pi^* = \int_0^L \left\{ \pi a \left[\frac{8D_s}{a^2}\Gamma_2^2 - \frac{18}{E_x t} f_2^2 - \frac{9a^2}{2Gt}(f_2')^2 - 2a^2\Gamma_2 f_2'' \right] - \left(\frac{M^2}{2E_x \bar{I}} + \frac{Q^2 \bar{R}}{2G} \right) \right\} \mathrm{d}x. \tag{29}$$

The equation $\delta\Pi^* = 0$ yields, after integration by parts:

$$\int_0^L \left\{ \left[\frac{16\pi D_s}{a}\Gamma_2 - 2\pi a^3 f_2'' - \frac{M^2}{2E_x}\frac{\partial}{\partial \Gamma_2}(1/\bar{I}) - \frac{Q^2}{2G}\frac{\partial}{\partial \Gamma_2}(\bar{R}) \right] \delta\Gamma_2 + \pi a \left(-\frac{36}{E_x t} f_2 + \frac{9a^2}{Gt} f_2'' - 2a^2 \Gamma_2'' \right) \delta f_2 - \sum_i \left(\frac{M}{E\bar{I}}\frac{\partial M}{\partial K_i} + \frac{Q\bar{R}}{G}\frac{\partial Q}{\partial K_i} \right) \delta K_i \right\} \mathrm{d}x$$

$$+ \pi a \left[\left(-\frac{9a^2}{Gt} f_2' + 2a^2 \Gamma_2' \right) \delta f_2 - 2a^2 \Gamma_2 \delta f_2' \right]_0^L = 0. \tag{30}$$

Equating to zero the coefficients of $\delta\Gamma_2$ and δf_2 in the integrand and on ∂L provides the equations and boundary conditions for the problem:

$$\left.\begin{aligned} &\frac{16\pi D_s}{a}\Gamma_2 - 2\pi a^3 f_2'' - \frac{M^2}{2E_x}\frac{\partial}{\partial \Gamma_2}(1/\bar{I}) - \frac{Q^2}{2G}\frac{\partial \bar{R}}{\partial \Gamma_2} = 0 \\ &-\frac{36}{E_x t} f_2 + \frac{9a^2}{Gt} f_2'' - 2a^2 \Gamma_2'' = 0 \\ &\int_0^L \left(\frac{M}{E_x \bar{I}}\frac{\partial M}{\partial K_i} + \frac{Q\bar{R}}{G}\frac{\partial Q}{\partial K_i} \right) \mathrm{d}x = 0 \quad (i = 1 \ldots n) \end{aligned}\right\} \text{ along } L \tag{31}$$

$$\left.\begin{aligned} &\left(-\frac{9f_2'}{Gt} + 2\Gamma_2' \right) \delta f_2 = 0 \\ &\Gamma_2 \delta f_2' = 0 \end{aligned}\right\} \text{ on } \partial L. \tag{32}$$

For the special case in which shear deformation is suppressed, the result is

$$\frac{16\pi D_s}{a}\Gamma_2 + \tfrac{1}{9}\pi a^5 E_x t \Gamma_2^{iv} = \frac{M^2}{2E_x I}\frac{\partial}{\partial \Gamma_2}(I/\bar{I})$$

$$f_2 = -\frac{a^2}{18} E_x t \Gamma_2''. \tag{33}$$

Boundary conditions: The *m*th Fourier coefficients of the normal and shearing stress resultants are expressed in terms of the stress function by

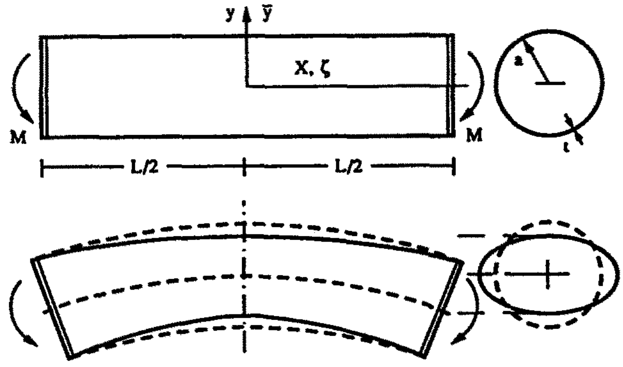

Fig. 1. Clamped tube subjected to a pure couple.

$$n_m^{xx} = -m(1-m^2)f_m, \quad n_m^{xs} = a(1-m^2)f'_m \quad (m > 1). \tag{34}$$

Also, the Gauss equation (with $e_{ss} = 0$) and stress–displacement relations of *linear* shell theory yield for the displacements v_x and v_s:

$$v_{x_m} = -\frac{a^2}{m(m^2-1)}\left[\frac{(m^2-1)^2}{Gt}f'_m + m\Gamma'_m\right] \quad v_{s_m} = -\frac{m}{m^2-1}\alpha\Gamma_m \quad (m > 1). \tag{35}$$

A comparison with the derived conditions above shows that while δf and $\delta f'$ represent the variations in n^{xx} and n^{xs}, respectively, their coefficients represent the *intrinsic equivalents* of the corresponding displacements. See discussion in the sections on general theory. The variational equation requires that either the stress resultant be specified (∂L_n condition) or else the corresponding displacement be zero (homogeneous condition on ∂L_v). The use of v_s and v_x is sufficient for small displacements, but the intrinsic equivalents retain their validity as boundary conditions even in the presence of larger displacements. For example, the specification of Γ determines the shape of the boundary curve (through k_{ss}), even for large deformations.

As noted before, semi-membrane theory does not have provisions for the specification of shell moments, transverse shears and their kinematical complements. This is a consequence of the suppression of the k_{xx}, k_{xs} and e_{ss} quantities in the energy. Edge effect corrections can be made at a later stage.

As a specific example, the case of an orthotropic tube of finite length subjected to a pure couple and attached to rigid rings at its boundaries, $x = 0, L$, will be studied.

3. EXAMPLE: ORTHOTROPIC TUBE CLAMPED TO RIGID RINGS AT ITS ENDS AND SUBJECTED TO PURE BEAM BENDING

An orthotropic tube of length L is attached to *rigid* rings or bulkheads at its ends $x = \pm L/2$, through which a couple M is applied. The effects of shear deformation are neglected ($\gamma_{xs} = 0$); see Figs 1 and 2.

The differential equation is cast in a nondimensional form by taking

$$\beta^4 = \frac{9}{4}\frac{D_s L^4}{E_x t a^6}; \quad q = \frac{a^4 t M^2}{32 I^2 E_x D_s}; \quad \zeta = \frac{2x}{L}; \quad \bar{\Pi}^* = \frac{a}{8\pi L D_s}\Pi^*. \tag{36}$$

The problem is symmetric with respect to its midsection $\zeta = 0$, so that the region $0 \leqslant \zeta \leqslant 1$ will be investigated. With these substitutions, the functional, differential equation and boundary conditions reduce, after elimination of f, to:

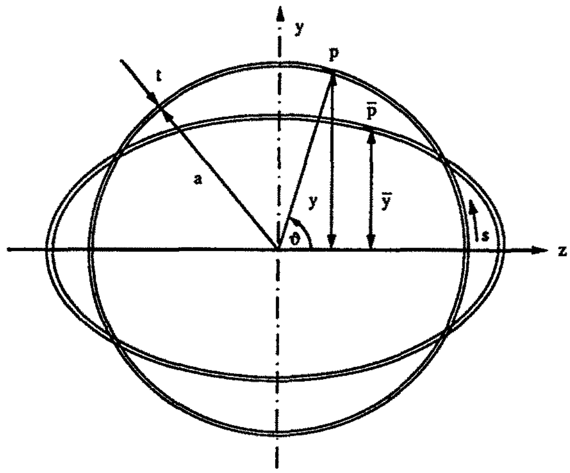

Fig. 2. The ovalization of a circular tube.

$$\tilde{\Pi}^* = \int_0^1 \left[\Gamma^2 + \frac{1}{4\beta^4} \Gamma_{,\zeta\zeta}^2 - 2q(I/\bar{I}) \right] \mathrm{d}\zeta$$

$$\Gamma + \frac{1}{4\beta^4} \Gamma_{,\zeta\zeta\zeta\zeta} = q \frac{\partial}{\partial \Gamma} (I/\bar{I})$$

$$\Gamma = \Gamma' = 0 \quad \text{at } \zeta = 1$$

$$\Gamma \text{ is symmetric with respect to } \zeta = 0. \tag{37}$$

Note that the subscript 2 was omitted for simplicity, so that $\Gamma \equiv \Gamma_2$. In the above, $I/\bar{I}$ is approximated to a degree consistent with the requirements. In particular, $I/\bar{I} \cong (1-2\Gamma)^{-1}$ (Brazier) or $I/\bar{I} \cong 1+2\Gamma+\frac{4}{9}\Gamma^2+\cdots$ (quadratic polynomial) may be used for this problem. The Brazier approximation will be used for the strongly nonlinear case, and the quadratic approximation for weaker nonlinearities which do not involve collapse analysis.

Based on the geometry or loading intensity, five distinct cases merit separate treatments.

(1) Long shells ($\beta^4 \gg 1$), away from the edges. This is the classical case. The deformation amplitude $\Gamma_\infty(q)$ is a solution of the equation

$$\Gamma = q \frac{\partial}{\partial \Gamma} (I/\bar{I}) \tag{38}$$

and analysis can be carried to collapse. Taking, for example, the Brazier approximation for $(I/\bar{I})$, Brazier's classic equilibrium path results:

$$q = \bar{q}_B(\Gamma) = (1/2)\Gamma(1-2\Gamma)^2. \tag{39}$$

A plot of $(\bar{q}_B)^{1/2}$ vs Γ is shown in Fig. 3 (marked with +).

For the case of orthotropic material, the above equation can be reduced to the following explicit expression for moment:

$$M = 4\pi a(E_x D_s t)^{1/2}(1-2\Gamma)\Gamma^{1/2}. \tag{40}$$

This expression agrees with the result of Kedward (1978).

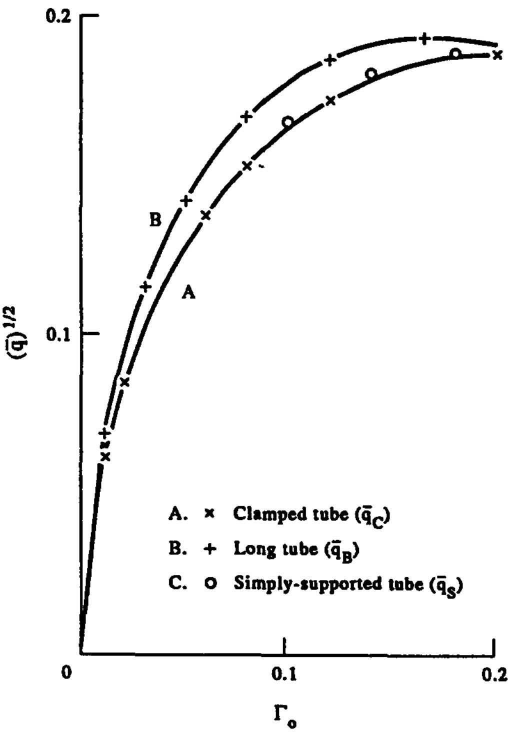

Fig. 3. Moment–deformation curves $\sqrt{\bar{q}}$–Γ_0 for several cases.

(2) Medium-length, lightly loaded shells $[\beta = O(1)]$. Since collapse analysis is not intended, the quadratic approximation for $I/\bar{I}$ may be used. Here†

$$\Gamma + (4\beta^4)^{-1}\Gamma_{,\zeta\zeta\zeta\zeta} = 2q(1+\tfrac{44}{9}\Gamma). \tag{41}$$

This is a linear equation of the "beam on elastic foundation" type. Its solution consists of a homogeneous part Γ_H which contains mixed hyperbolic–trigonometric functions, and a particular solution Γ_p which is a constant. To improve accuracy, $\Gamma_p = \Gamma_\infty$ can be taken. For the boundary and symmetry conditions at hand, the result is:

$$\Gamma = \{1-2(\sinh 2\bar{\beta}+\sin 2\bar{\beta})^{-1}[(\cosh\bar{\beta}\sin\bar{\beta}+\sinh\bar{\beta}\cos\bar{\beta})\cosh\bar{\beta}\zeta\cos\bar{\beta}\zeta$$
$$+(\cosh\bar{\beta}\sin\bar{\beta}-\sinh\bar{\beta}\cos\bar{\beta})\sinh\bar{\beta}\zeta\sin\bar{\beta}\zeta]\}\Gamma_\infty, \tag{42}$$

where

$$(\bar{\beta})^4 = (1-\tfrac{88}{9}q)\beta^4.$$

The effects of the finite length of the shell are contained entirely in $\bar{\beta}$. Although eqn (42) is valid for all $\bar{\beta}$, it is best used for $0.5 < \bar{\beta} < 3$. For larger $\bar{\beta}$, the solution merges into an edge

† To be consistent with the Brazier approximation, replace 44/9 with 4.

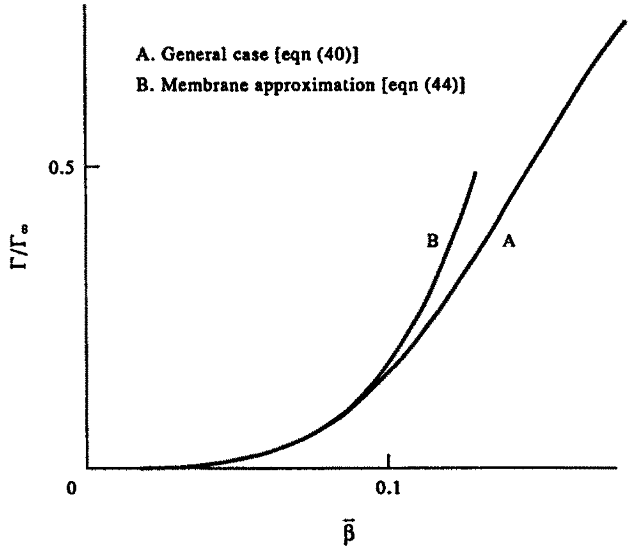

Fig. 4. Clamped tube: amplitude ratio at midspan (weak nonlinearity).

effect response (case 3) and for smaller $\bar{\beta}$ the membrane response (case 4) dominates; see also Fig. 4.

(3) Long shells—edge effects. Edge effects in a long shell decay into the interior, where the classical solution dominates. The quadratic approximation is used, which makes it a special case of (2). Here, x (measured from the edge!) is used as a variable. The solution is

$$\Gamma = [1-e^{-\mu x}(\cos\mu x+\sin\mu x)]\Gamma_\infty, \tag{43}$$

where $\mu = 2\bar{\beta}/L$ is a decay parameter, which increases with q.

(4) Short shells ($\beta^4 \ll 1$). Here, the first term in the equation may be neglected. Since this term represents the shell bending effects, this results in the *nonlinear membrane formulation* for the tube problem (note that only geometric nonlinearity is considered):

$$\Gamma_{,\zeta\zeta\zeta\zeta} = 4\beta^4 q \frac{\partial}{\partial\Gamma}(I/\bar{I}). \tag{44}$$

For short shells, the deformation amplitude is small up to buckling, and the linear terms in the expansion of $I/\bar{I}$ are sufficient, yielding

$$\Gamma_{,\zeta\zeta\zeta\zeta} = 8\beta^4 q. \tag{45}$$

The solution which satisfies the conditions at $\zeta = 0, 1$ is

$$\Gamma = \Gamma_0(1-\zeta^2)^2 \quad \text{with } \Gamma_0 = \tfrac{1}{3}\beta^4 q. \tag{46}$$

Good results are obtained for $\beta < 0.5$. In these short shells, the main resistance to ovalization is due to membrane resultants n^{xx} and n^{xs} stemming from the supporting rings. This solution also provides a simple and convenient "shape function" for the large deformation and collapse analysis of medium-length shells, using Rayleigh–Ritz procedures (see next case).

The solution of case (2) degenerates into the short-shell solution as β becomes smaller.

(5) High-intensity loading, large deformations, and collapse of medium-length shells. Here the quadratic approximation for $I/\bar{I}$ is insufficient and the denominator form is retained. The Brazier form, $(1-2\Gamma)^{-1}$, is adequate, since it has been shown to yield good

results for large deformations. See Reissner and Weinitschke (1963). A one-term Rayleigh procedure is employed for the approximate solution of the variational equation $\delta\Pi^* = 0$, with the "shape function" taken from the weakly nonlinear solution. This is a common procedure used in many branches of applied science for obtaining approximate solutions to difficult, highly nonlinear problems. It is reasonable in stable problems in which the shape is well chosen and is approximately preserved in the nonlinear range, as is the case here.

Let $g(\zeta)$ be an assumed shape function which satisfies all of the boundary conditions, such that $\Gamma = \Gamma_0 g(\zeta)$. The parameter to be varied is Γ_0 but g may also contain other parameters (such as β and q). Let

$$\eta(\beta) = \frac{1}{4\beta^4}\left[\int_0^1 g_{,\zeta\zeta}^2\,\mathrm{d}\zeta\right]\left[\int_0^1 g^2\,\mathrm{d}\zeta\right]^{-1} \tag{47}$$

be the ratio of extensional to bending energies for the assumed Γ. Then, the equation $\delta\Pi^* = 0$ reduces to

$$\int_0^1 \left[\Gamma - \bar{q}\frac{\partial}{\partial\Gamma}(I/\bar{I})\right]\delta\Gamma\,\mathrm{d}\zeta = 0. \tag{48}$$

With $\bar{I}/I = 1-2\Gamma$, $\Gamma = \Gamma_0 g$ and $\bar{q}$ defined by $\bar{q} = (1+\eta)^{-1}q$, this yields

$$\bar{q} = (\Gamma_0/2)\int_0^1 g^2\,\mathrm{d}\zeta\left[\frac{\int_0^1 g\,\mathrm{d}\zeta}{(1-2\Gamma_0 g)^2}\right]^{-1}. \tag{49}$$

Since the applied bending moment is proportional to $q^{1/2}$, the function $q^{1/2} = f(\Gamma_0, \beta)$ describes the moment–deformation response up to collapse or buckling (which precedes collapse—see subsequent note).

The best suited choice for g is that of eqn (42), which is valid for all β. It is, however, rather elaborate, and simplifications are welcome where appropriate.

For short- to medium-length shells (say $\beta < 0.8$), the short-shell form $g = (1-\zeta^2)^2$ can be used. [A similar choice made by Libai and Bert (1991) for the simply supported case gave results which deviated only slightly from those of the more general choice.] Here, $\eta = (63/8)\beta^{-4}$. Denoting by $\bar{q}_c(\Gamma_0)$ the form of eqn (49) for this case, one obtains the plot given in Fig. 3 (marked with symbol X) as curve A. Also shown (marked with circles) is $\bar{q}_s(\Gamma_0)$ for the simply supported case, obtained by Libai and Bert (1991), and with $\eta\beta^4 = 1.52$. It appears that $\bar{q}_s(\Gamma_0)$ is virtually identical to $\bar{q}_c(\Gamma_0)$.

For medium-length to long shells (say, $\beta > 3$), the choice is influenced by the realization that the mechanism of highly nonlinear response and collapse far from the edges of a longer tube is local and circumferential in nature. It is influenced only slightly by variations in the x-direction. Furthermore, eqn (48) bears a formal resemblance to that of infinitely long shells, eqn (38). Hence, for this case, it makes sense to choose the long-shell response, $g(\zeta) \equiv 1$, which yields the Brazier curve $\bar{q}_B(\Gamma_0)$. On the other hand, the membrane contribution is an integral effect which stems from the supports, so that the energy ratio of the shorter shells is still adequate. It is noted that the maximum of $\bar{q}_c^{1/2}$ is 0.187 at $\Gamma_0 = 0.198$. This is 2.6% lower than Brazier's collapse value of $(\bar{q}_B)_{\max}^{1/2} = 0.192$, but the slight difference is more than offset by the effects of β. The fact that the short-shell curve slightly "underestimates" the long-shell response is not surprising in view of the different shape functions involved and the complementary nature of the membrane terms in the variational principle.

For medium-length shells ($0.8 \leqslant \beta \leqslant 3$), the response is intermediate between $\bar{q}_B$ and $\bar{q}_c$. However, in view of the slight differences between the two, an interpolation should provide adequate results. For better accuracy, eqn (42) can be used.

A note on buckling and collapse: it should be emphasized that bent tubes fail by local buckling on their compressive sides before the classical Brazier collapse is achieved. In very

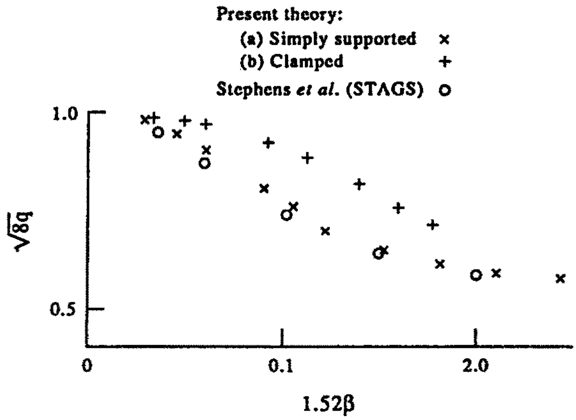

Fig. 5. Local buckling of a bent tube.

long shells subjected to pure bending, buckling is close enough to collapse to allow the latter to be used as a substitute failure point. In shorter shells, buckling occurs significantly below collapse, but the nonlinear analysis provides the necessary data for local buckling analysis. See Axelrad (1985b) for more details. In very short shells, L/a of the order $O(1)$, bending nonlinearity barely develops before buckling occurs, and it does not have a significant effect.

Results of local buckling analysis based on the present theory are compared with nonlinear buckling results obtained by Stephens *et al.* (1975) using the STAGS computer program. The emphasis is on shell length effects. Considering the approximations that went into the present theory and the fact that the boundary conditions did not match exactly, the comparison is very satisfactory and can be used as an implicit numerical check on the theory; see Fig. 5.

4. SUMMARY AND REMARKS

A mixed variational principle for nonlinear shell problems was developed. It is based on a "curvature function" ψ, and is therefore limited to those shells where ψ can be obtained. For these cases, it can accommodate deformations where the strains are small but the rotations are unrestricted in magnitude. More details were given for shells of weak curvature and for cylindrical shells, and indications for obtaining ψ in some other cases were included.

The development of the principle for the bending of finite-length orthotropic tubes was given in more detail, and stronger nonlinearities were incorporated in the formulation. Differential equations and boundary conditions were developed for an approximate engineering approach to the problem, and the case of a clamped shell subjected to a pure couple was solved. Some of the results compare favorably with numerical local buckling data from the literature.

Further work is needed in this area. Some problem areas are:

(a) The extension to other shell geometries.
(b) Consideration of boundary conditions of more general types including closure conditions in multiply-connected shells. See also Sanders (1970) and Valid (1976).
(c) Incorporation of surface loads. Can these be incorporated as particular solutions of membrane equilibrium?
(d) Considerations of buckling and postbuckling phenomena. In this respect, the reincorporation of the *nonlinear strain gradient terms* [second line in Danielson (1970) or equivalent terms in other quoted literature] into the Gauss equation L_3 in Part I, eqns (2) and (17), should be considered. See also the remark in the paragraph preceding Section 2.1 of Part I. In fact, the proper introduction of these

terms into eqn (3.15) of Danielson (1970) would render his system of eqns (3.15)–(3.18) (or equivalent systems in the other quoted literature) *variationally consistent*—in the sense that it would be derivable from a variational principle.
(e) Extensions to nonlinear dynamics.
(f) Consideration of other material systems such as anisotropic elastic materials and elastoplastic materials.

In tube problems, in addition to the above, more solutions are needed, and stronger nonlinearities need to be incorporated. In particular, in view of the scarcity of data in this range of deformation, critical experiments and numerical comparisons are necessary.

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